

From 4.2

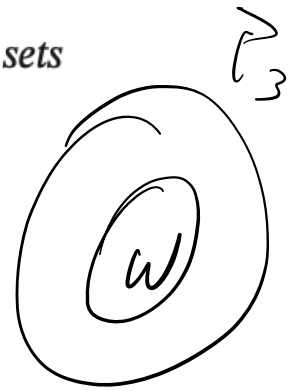
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i) If $\vec{u}, \vec{v} \in W$ then $\vec{u} + \vec{v} \in W$

In Exercises 5-6, use the Subspace Test to determine which of the sets are subspaces of P_3 .

5. a. All polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 = 0$.

b. All polynomials $a_0 + a_1x + a_2x^2 + a_3x^3$ for which $a_0 + a_1 + a_2 + a_3 = 0$.



a. Let $\vec{u} = u_1x + u_2x^2 + u_3x^3$ and

$$\vec{v} = v_1x + v_2x^2 + v_3x^3 \in W$$

$$\vec{u} + \vec{v} = (u_1 + v_1)x + (u_2 + v_2)x^2 + (u_3 + v_3)x^3, \text{ which}$$

has constant term 0.

$$k\vec{u} = ku_1x + ku_2x^2 + ku_3x^3, \text{ which again has}$$

constant 0.

b. Let $\vec{u} = u_0 + u_1x + u_2x^2 + u_3x^3$ and

$$\vec{v} = v_0 + v_1x + v_2x^2 + v_3x^3 \in P_3$$

Then $k\vec{u} + \vec{v} = (ku_0 + v_0) + \dots + (ku_3 + v_3)x^3$

$$= (ku_0 + v_0) + (ku_1 + v_1) + (ku_2 + v_2) + (ku_3 + v_3)$$

$$= k(u_0 + \dots + u_3) + v_0 + \dots + v_3$$

$$= 0 + 0 = 0$$

8.1 – General Linear Transformations

Definition: (analogous to Theorem 1.8.2)

If $T : V \rightarrow W$ is a mapping from a vector space V to a vector space W , then T is called a **linear transformation** from V to W if the following two properties hold for all vectors \mathbf{u} and \mathbf{v} in V and for all scalars k :

- $T(k\mathbf{u}) = kT(\mathbf{u})$ (homogeneity property)
- $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ (additivity property)

In the special case where $V = W$, the linear transformation T is called a **linear operator** on the vector space V .

Theorem 8.1.1 (analogous to Theorem 1.8.1)

If $T : V \rightarrow W$ is a linear transformation, then:

- $T(\mathbf{0}) = \mathbf{0}$.
- $T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v})$ for all \mathbf{u} and \mathbf{v} in V .
- $T(-\mathbf{v}) = -T(\mathbf{v})$ for all \mathbf{v} in V .

What if $T(\vec{0}) \neq \vec{0}$?

T is not linear.

pf (a): Since T is linear, $T(k\vec{v}) = kT(\vec{v})$

for all scalars k . Let $k = 0$.

$$T(\mathbf{0}) = T(0\vec{v}) = 0T(\vec{v}) = \mathbf{0} \quad \checkmark$$

Examples of Linear transformations

- $T : V \rightarrow W$ defined by $T(\mathbf{v}) = \mathbf{0}$ for all $\mathbf{v} \in V$ (the zero transformation)
- $I : V \rightarrow V$ defined by $I(\mathbf{v}) = \mathbf{v}$ (the identity operator)
- $T : V \rightarrow V$ defined by $T(\mathbf{x}) = c\mathbf{x}$ (contraction if $0 < c < 1$ and dilation if $c > 1$)
- $T : V \rightarrow R$ defined by $T(\mathbf{x}) = \langle \mathbf{x}, \mathbf{x}_0 \rangle$ (inner product of \mathbf{x} with \mathbf{x}_0) [We'll see this in Ch. 6]
- $T : M_{nn} \rightarrow M_{nn}$ defined by $T(A) = A^T$

x	$f(x)$
a	$f(a)$
b	$f(b)$
c	$f(c)$
d	$f(d)$

- $T : V \rightarrow R^n$ defined by $T(f) = (f(x_1), f(x_2), \dots, f(x_n))$, where V is a subspace of $F(-\infty, \infty)$, and x_1, x_2, \dots, x_n is a sequence of real numbers (evaluation transformation)
- $D : C^1(-\infty, \infty) \rightarrow F(-\infty, \infty)$ defined by $D(f) = f'(x)$ (differentiation)
- $J : C(-\infty, \infty) \rightarrow C^1(-\infty, \infty)$ defined by $J(f) = \int_0^x f(t) dt$ (integration)

Theorem 8.1.2 Let $T : V \rightarrow W$ be a linear transformation, for which the vector space V is finite-dimensional. If $S = \{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\}$ is a basis for V , then the image of any vector \mathbf{v} in V can be expressed as

$T(\mathbf{v}) = c_1 T(\mathbf{v}_1) + c_2 T(\mathbf{v}_2) + \dots + c_n T(\mathbf{v}_n)$ where c_1, c_2, \dots, c_n are the coefficients required to express \mathbf{v} as a linear combination of the vectors in the basis S .

pf: Let $\vec{v} \in V$. Then since S is a basis, $\exists c_i \ni \vec{v} = \sum_{i=1}^n c_i \vec{v}_i$. $T(\vec{v}) = T\left(\sum_{i=1}^n c_i \vec{v}_i\right)$
 $= \sum_{i=1}^n T(c_i \vec{v}_i) = \sum_{i=1}^n c_i T(\vec{v}_i)$ since T is linear.

$T(c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_n \vec{v}_n) = T(c_1 \vec{v}_1) + \dots + T(c_n \vec{v}_n) = c_1 T(\vec{v}_1) + \dots + c_n T(\vec{v}_n)$
 #20 Consider the basis $S = \{\mathbf{v}_1, \mathbf{v}_2\}$ for R^2 , where $\mathbf{v}_1 = (-2, 1)$ and $\mathbf{v}_2 = (1, 3)$, and let $T : R^2 \rightarrow R^3$ be the linear transformation such that $T(\mathbf{v}_1) = (-1, 2, 0)$ and $T(\mathbf{v}_2) = (0, -3, 5)$. Find a formula for $T(x_1, x_2)$ and use that formula to find $T(2, -3)$.

When we did this in 1.8, we wanted the standard matrix for the transformation. Here we want the formula for T .

For example, $T(x_1, x_2) = (2x_1 + 3x_2, 2x_2 - x_1)$

If $\vec{v} = c_1 \vec{v}_1 + c_2 \vec{v}_2$, then $T(\vec{v}) = c_1 T(\vec{v}_1) + c_2 T(\vec{v}_2)$

We start by expressing $\vec{x} = (x_1, x_2)$ as a linear combination of \vec{v}_1 & \vec{v}_2 :

that is, we want c_1, c_2 such that

$$c_1(-2, 1) + c_2(1, 3) = (x_1, x_2)$$

$$\left[\begin{array}{cc|c} -2 & 1 & x_1 \\ 1 & 3 & x_2 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & -\frac{3}{7}x_1 + \frac{1}{7}x_2 \\ 0 & 1 & \frac{1}{7}x_1 + \frac{2}{7}x_2 \end{array} \right]$$

$$T(\vec{x}) = \left(-\frac{3}{7}x_1 + \frac{1}{7}x_2\right)(-1, 2, 0) + \left(\frac{1}{7}x_1 + \frac{2}{7}x_2\right)(0, -3, 5)$$

$$T(\vec{x}) = \left(\frac{3}{7}x_1 - \frac{1}{7}x_2, -\frac{6}{7}x_1 + \frac{2}{7}x_2, 0\right) + \left(0, -\frac{3}{7}x_1 - \frac{6}{7}x_2, \frac{5}{7}x_1 + \frac{10}{7}x_2\right)$$

$$T(x_1, x_2) = \left(\frac{3}{7}x_1 - \frac{1}{7}x_2, -\frac{9}{7}x_1 - \frac{4}{7}x_2, \frac{5}{7}x_1 + \frac{10}{7}x_2\right)$$

$$T(2, -3) = \left(\frac{6}{7} + \frac{3}{7}, -\frac{18}{7} + \frac{12}{7}, \frac{10}{7} - \frac{30}{7}\right)$$

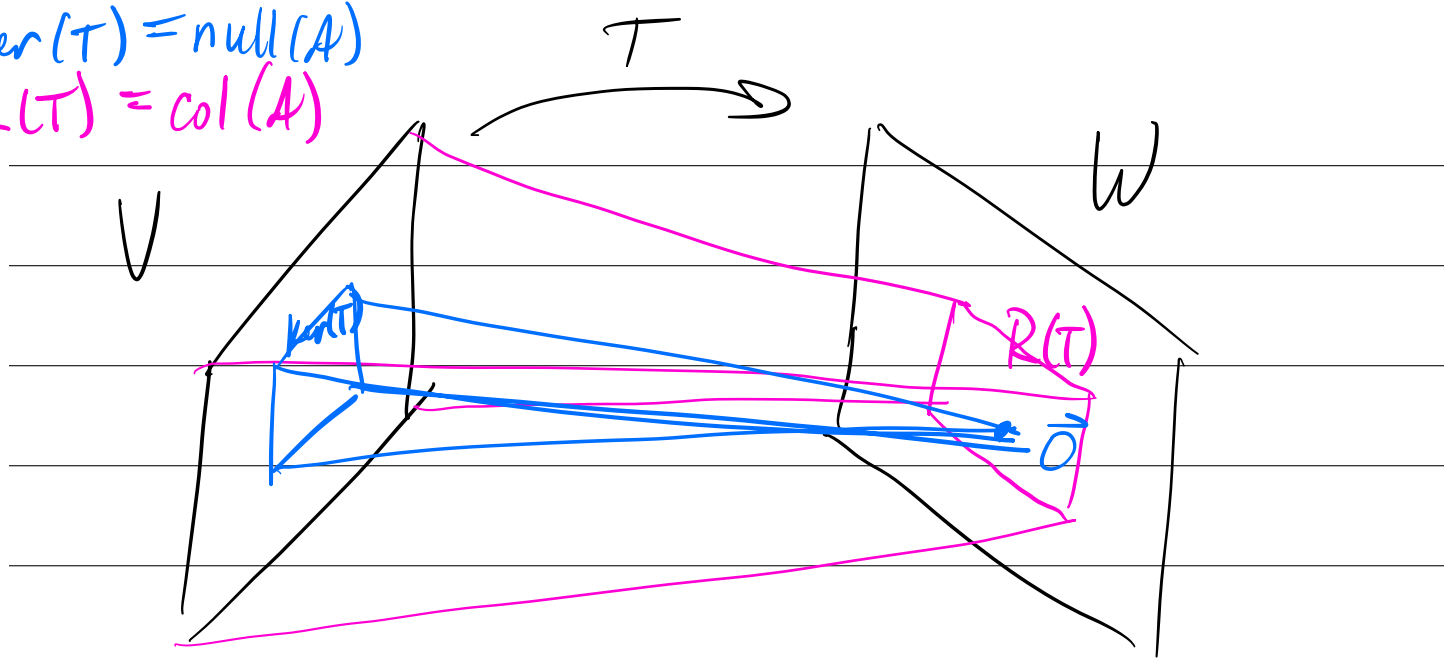
$$T(2, -3) = \left(\frac{9}{7}, -\frac{6}{7}, -\frac{20}{7}\right)$$

Definition: (analogous to definitions seen in Section 4.2 (**kernel**) and Section 1.8 (**range**); also related to Definition 2 of Section 4.8 (**null space**) and, by Theorem 4.8.1, column space)

If $T : V \rightarrow W$ is a linear transformation, then the set of vectors in V that T maps into $\mathbf{0}$ is called the **kernel** of T and is denoted by $\ker(T)$. The set of all vectors in W that are images under T of at least one vector in V is called the **range** of T and is denoted by $R(T)$.

$$\ker(T) = \text{null}(A)$$

$$R(T) = \text{col}(A)$$



#6 Determine whether the mapping T is a linear transformation, and if so, find its kernel.

$T : M_{22} \rightarrow R$, where

a. $T \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = 3a - 4b + c - d$

b. $T \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right) = a^2 + b^2$

NO

$$b) T \left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \right) = T \left(\begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix} \right) = (a_1+a_2)^2 + (b_1+b_2)^2$$

$$= a_1^2 + 2a_1a_2 + a_2^2 + b_1^2 + 2b_1b_2 + b_2^2$$

$$\neq a_1^2 + b_1^2 + a_2^2 + b_2^2$$

OR

$$T(k \begin{bmatrix} a & b \\ c & d \end{bmatrix}) = T \left(\begin{bmatrix} ka & kb \\ kc & kd \end{bmatrix} \right) = k^2 a^2 + k^2 b^2 = k^2 (a^2 + b^2) \neq k T \left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right)$$

Streamlined:

$$\begin{aligned}
 \text{a) } T\left(k \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) &= T\left(\begin{bmatrix} ka_1 + a_2 & kb_1 + b_2 \\ kc_1 + c_2 & kd_1 + d_2 \end{bmatrix}\right) \\
 &= 3(ka_1 + a_2) - 4(kb_1 + b_2) + (kc_1 + c_2) - (kd_1 + d_2) \\
 &= k(3a_1 - 4b_1 + c_1 - d_1) + (3a_2 - 4b_2 + c_2 - d_2) \\
 &= k T\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) \text{ this is linear.}
 \end{aligned}$$

$$\ker(T) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mid 3a - 4b + c - d = 0 \right\}$$

#10 Let $T : P_2 \rightarrow P_3$ be the linear transformation defined by $T(p(x)) = xp(x)$.

Which of the following are in $\ker(T)$?

a. x^2

b. 0

c. $1 + x$

d. $-x$

$$\left\{ p(x) \in P_2 \mid xp(x) = 0 \right\}$$

#11 Let $T : P_2 \rightarrow P_3$ be the linear transformation in Exercise 10. Which of the following are in $R(T)$?

$$T(p(x)) = xp(x)$$

a. $x + x^2$

b. $1 + x$

c. $3 - x^2$

d. $-x$

Range is polynomials of $\deg \leq 3$ with constant term 0.

#25 Let $T_A : R^4 \rightarrow R^3$ be multiplication by A . Find a basis for the kernel of T_A , and then find a basis for the range of T_A that consists of column vectors of A .

$$A = \begin{bmatrix} 1 & 2 & -1 & -2 \\ -3 & 1 & 3 & 4 \\ -3 & 8 & 4 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & -10/7 \\ 0 & 1 & 0 & -2/7 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad \begin{array}{l} x_1 = \frac{10}{7}x_4 \\ x_2 = \frac{2}{7}x_4 \\ x_3 = 0 \end{array} \quad \text{Let } 7t = x_4$$

$$\vec{x} = \begin{bmatrix} 10t \\ 2t \\ 0 \\ 7t \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \\ 0 \\ 7 \end{bmatrix} t$$

Basis for $\ker(T)$ is $\left\{ \begin{bmatrix} 10 \\ 2 \\ 0 \\ 7 \end{bmatrix} \right\}$.

Pivot columns are 1, 2, 3, so a basis for $\mathcal{R}(T)$ is $\left\{ \begin{bmatrix} 1 \\ -3 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 8 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 4 \end{bmatrix} \right\}$.

Theorem 8.1.3 If $T : V \rightarrow W$ is a linear transformation, then:

- The kernel of T is a subspace of V .
- The range of T is a subspace of W .

Definition: (analogous to Definition 1 of Section 4.9)

Let $T : V \rightarrow W$ be a linear transformation. In the case that the range of T is finite-dimensional its dimension is called the **rank of T**, denoted by $\text{rank}(T)$; and if the kernel of T is finite-dimensional, then its dimension is called the **nullity of T**, denoted by $\text{nullity}(T)$.

Theorem 8.1.4 Dimension Theorem for Linear Transformations

(generalization of Theorem 4.9.2)

If $T : V \rightarrow W$ is a linear transformation from a finite-dimensional vector space V to a vector space W , then the range of T is finite-dimensional, and

$$\text{rank}(T) + \text{nullity}(T) = \text{dim}(V). \text{ domain}$$

#13 In each part, use the given information to find the nullity of the linear transformation T .

a. $T : R^5 \rightarrow P_5$ has rank 3. $\text{dim}(R^5) = 5$ $\text{dim}(P_5) = 6$

b. $T : P_4 \rightarrow P_3$ has rank 1. $\text{dim}(P_4) = 5$ $\text{dim}(P_3) = 4$

c. The range of $T : M_{mn} \rightarrow R^3$ is R^3 .

d. $T : M_{22} \rightarrow M_{22}$ has rank 3.

$$\text{dim}(\text{ker}(T))$$

$$a) 5 - 3 = 2 = \text{nullity}(T)$$

$$b) 5 - 1 = 4 = \text{nullity}(T)$$

$$c) mn - 3 = \text{nullity}(T)$$

$$d) 4 - 3 = 1 = \text{nullity}(T)$$